Enhancements to Worst-Case Performance Assessment Calculations for Uncertain Linear Systems

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We present a few elementary results and observations aimed at simplifying the worst-case performance assessment of linear, dynamic systems subjected to real-parameter and unmodeled dynamics uncertainty. We focus on a constant-matrix problem, which through state-space models, and/or frequency domain techniques, extends to treatment of dynamic systems. We follow the now “standard” technique for assessing worst-case performance – computable lower and upper bounds which are refined through divide-and-conquer strategies, addressing 3 topics:

1. A simple algorithm to compute lower bounds for worst-case performance problems is described. It mimics Hamiltonian methods for state-space norm calculations. Results from computational experiments are given, and comparisons to other approaches are made.

2. Initial feasibility of the well-known LMI upper bound conditions in subdivided parameter cubes (arising in a Divide-and-Conquer approach) is proven. Exploiting this typically reduces computation time by 20-35%.

3. Transformations which allow specific types of correlations between pairs of parameters are given.

The methodology is tested on atmospheric models of the X-38 Crew Recovery Vehicle used in recent drop tests.

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• Worst-Case Performance for LFTs
• Divide and Conquer, based on upper and lower bounds
• Lower Bound
• Upper Bound
• Correlating two uncertain parameters
• Results
Systems under consideration:

- Linear systems with
  - parametric uncertainty, and/or
  - unmodelled dynamics

- Performance objective involves keeping specific transfer functions “small”

Uncertain relationship between $d$ and $e$ is

\[
e = [M_{22} + M_{21} \Delta (I - M_{11} \Delta) M_{12}] d
\]

\[
=: F_u(M, \Delta) d
\]

\[
=: T_{d\rightarrow e}(\Delta) d
\]

In this diagram, the known elements are separated from unknown elements in this feedback connection.

- Known contains: nominal plant model, controller, manner in which uncertainty enters, disturbances, errors

- Unknown contains: uncertainty in parameters of differential equations, unmodeled dynamics
Worst-Case Performance

Why not robust stability?

- Before stability is lost, performance degrades unacceptably
- Worst-Case Performance, over parameter uncertainties and un-modelled dynamics, is easy to motivate, and uses same mathematical tools as Robust Stability calculations
- For problems with only real parametric uncertainty modeled, Robust Stability quantities can be discontinuous (in data and frequency)
**Justification**

**Question:** is the quantity

\[
\max_{\text{allowable}} \Delta \max_{\omega} |T_{d\rightarrow e}(\Delta, \omega)|
\]

a good measure of “worst-case behavior?”

- Honeywell SRC applied this type of analysis to Shuttle in 1984+.
- Frequency domain criterion – connection to time domain is less precise than usually desired
- \( T \) must be chosen carefully to reflect variables of interest
- One strategy: assess and normalize the nominal level of performance being achieved by current controller
One method to pick $T$:

- Start with nominal model, and candidate controller
- Plot closed-loop frequency response from commands ($r$) and gusts ($g$) to tracking error $e$

\[
e = G^{\text{nom}} \begin{bmatrix} r \\ g \end{bmatrix}
\]

- Find simple weighting functions $W_1, W_2$ such that

\[
|W_1(j\omega)G_1^{\text{nom}}(j\omega)| \approx 1, \quad |W_2(j\omega)G_2^{\text{nom}}(j\omega)| \approx 1
\]

for all frequencies
- Hence, the nominal model with controller achieves weighted closed-loop performance

\[
\max_{\omega} \left\| \frac{W_1(j\omega)G_1^{\text{nom}}(j\omega)}{W_2(j\omega)G_2^{\text{nom}}(j\omega)} \right\| \approx 1.4
\]

- For perturbed system, use this objective, and see how “bad” it can be made, relative to 1.4 (nominal performance)

\[
T(\Delta) := \begin{bmatrix} W_1G_1(\Delta) \\ W_2G_2(\Delta) \end{bmatrix}
\]

- Find

\[
\max_{\Delta \text{ allowable}} \| T(\Delta) \|_{\infty}
\]
Treat two types of model uncertainty:

1. Uncertain real-valued parameters in differential equation model
2. Unmodeled dynamics, with frequency dependent bounds

Normalizing (absorbing offsets and weights into “known” part of system) yields uncertain matrices of the form

$$\Delta = \text{diag} [\delta_1 I_{k_1}, \ldots, \delta_n I_{k_n}, \hat{\Delta}_1(s), \ldots, \hat{\Delta}_f(s)]$$

with each real parameter and transfer function parameter assumed to satisfy

$$|\delta_i| \leq 1, \quad \max_\omega |\hat{\Delta}_i(j\omega)| \leq 1$$

**Easy Fact:** Given any complex number $\gamma$, with $|\gamma| \leq 1$, and any frequency $\omega_0 > 0$, there is a stable transfer function $\hat{\Delta}(s)$ satisfying

$$\max_\omega |\hat{\Delta}(j\omega)| = |\gamma|, \quad \hat{\Delta}(j\omega_0) = \gamma$$

**Implication:** Ultimately treat the Worst-Case-Performance computation as a collection of uncoupled constant matrix problems.
Reduction to Constant-Matrix

For the problem, there is a “worst” frequency. At that frequency, the mathematical problem is a single matrix problem. How? Allowable \( \Delta \) satisfy

\[
\Delta = \text{diag} \left[ \delta_1 I_{k_1}, \ldots, \delta_n I_{k_n}, \hat{\Delta}_1(s), \ldots, \hat{\Delta}_f(s) \right]
\]

with

\[
|\delta_i| \leq 1, \quad \max_\omega |\hat{\Delta}_i(j\omega)| \leq 1
\]

Original problem is

\[
\max_{\Delta(s) \text{ allowable}} \max_\omega \left| F_u \left( \hat{M}(j\omega), \Delta(j\omega) \right) \right|
\]

Interchange the “\( \max \)”

\[
\max_\omega \max_{\Delta(s) \text{ allowable}} \left| F_u \left( \hat{M}(j\omega), \Delta(j\omega) \right) \right|
\]

But at any fixed frequency, the transfer function entries of \( \Delta(j\omega) \) can be any complex number with magnitude \( \leq 1 \). So, at each frequency, view \( \Delta \) as a constant matrix (real and complex entries)

\[
\max_\omega \max_\Delta \left| F_u \left( \hat{M}(j\omega), \Delta \right) \right|
\]

Grid frequency range, based on domain-specific expertise, with finite frequencies, \( \omega_1, \omega_2, \ldots, \omega_N \), and solve only there

\[
\max_{1 \leq i \leq N} \max_\Delta \left| F_u \left( \hat{M}(j\omega_i), \Delta \right) \right|
\]
Focus on constant matrix problem (and solve at many frequencies)

- \( M \in \mathbb{C}^{(k+n_e)\times(k+n_d)} \), complex since this will typically be response of \( \hat{M} \) at a certain frequency.
- Integers \( k_1, \ldots, k_n, k_{n+1}, \ldots, k_{n+f} \), with \( k := k_1 + \cdots + k_{n+f} \).
- \( n \) uncertain real parameters, \( \delta_1, \ldots, \delta_n \), each varies independently in range, \( a_i \leq \delta_i \leq b_i \)
- \( f \) uncertain matrices, \( \Delta_1 \in \mathbb{C}^{k_{n+1} \times k_{n+1}}, \ldots, \Delta_f \in \mathbb{C}^{k_{n+f} \times k_{n+f}} \)
- Associated with the indices \( k_i \), \( \mathcal{D} \) denotes the operation which takes \( \delta := (\delta_1, \ldots, \delta_n) \) and \( \Delta := (\Delta_1, \ldots, \Delta_f) \) into the \( k \times k \) block-diagonal matrix

\[
\mathcal{D}_{\delta\Delta} := \text{diag} [\delta_1 I_{k_1}, \ldots, \delta_n I_{k_n}, \Delta_1, \ldots, \Delta_f]
\]

**Problem:** Given \( M \) and the intervals \([a_i b_i]\), estimate lower and upper bounds for

\[
\max_{a_i \leq \delta_i \leq b_i, \quad \sigma(\Delta_i) \leq 1} \bar{\sigma} \left[ M_{22} + M_{21} \mathcal{D}_{\delta\Delta} (I - M_{11} \mathcal{D}_{\delta\Delta})^{-1} M_{12} \right]
\]
Divide-and-Conquer only on Real parameters

Given $M$, $a$ and $b$. Need easily computable bounds $L$ and $U$

- Lower bound, $L(a, b, M)$, satisfying

$$L(a, b, M) \leq \max_{\substack{a_i \leq \delta_i \leq b_i \\ \delta(\Delta_i) \leq 1}} \bar{\sigma} [M_{22} + M_{21} \mathcal{D}_{\delta\Delta} (I - M_{11} \mathcal{D}_{\delta\Delta})^{-1} M_{12}]$$

- Upper bound, $U(a, b, M)$, satisfying

$$\max_{\substack{a_i \leq \delta_i \leq b_i \\ \delta(\Delta_i) \leq 1}} \bar{\sigma} [M_{22} + M_{21} \mathcal{D}_{\delta\Delta} (I - M_{11} \mathcal{D}_{\delta\Delta})^{-1} M_{12}] \leq U(a, b, M)$$

Bounds $L$ and $U$ presented today have property that for $a = b$, $L$ is reasonably close to $U$. For $a < b$, gap increases, and a “Divide-and-Conquer” reduces the gap.

Steps of Divide & Conquer

1. Initialize list of cubes to the initial cube,

$$[a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$$

2. Call upper and lower bounds computations on the initial cube.

3. Find cube in ACTIVE list with largest upper bound

4. Split cube along longest edge into two cubes, compute bounds on both of these new cubes, and replace

5. Make any current cube whose upper bound is lower than another cube’s lower bound INACTIVE. Go to 3

We do not divide on complex \{\Delta_i\}. Simply accept gap in $L$ and $U$ that exists even when $a = b$. 
Divide-and-Conquer

One Dimensional Overview

- One variable ($\delta I_4$), Solid - active; dashed - inactive
**Problem:** Given $M$ and the intervals $[a_i b_i]$, estimate lower and upper bounds for

$$\max_{a_i \leq \delta_i \leq b_i, \bar{\sigma}(\Delta_i) \leq 1} \bar{\sigma} \left[ M_{22} + M_{21} D_{\delta \Delta} (I - M_{11} D_{\delta \Delta})^{-1} M_{12} \right]$$

drawn as

![Diagram]

Recall,

$$D_{\delta \Delta} := \text{diag} [\delta_1 I_{k_1}, \ldots, \delta_n I_{k_n}, \Delta_1, \ldots, \Delta_f]$$

Strategy – coordinatewise across $\delta$ and $\Delta$, but with different approaches:

1. Hold complex uncertainties ($\Delta_j$) fixed, maximize over real parameters using coordinate-wise ascent

2. Holding real parameters ($\delta_i$) fixed, maximize over complex uncertainties using power method

3. Iterate.

The individual steps are as follows...
Given $M$, intervals $[a_1, b_1], \ldots, [a_n, b_n]$, estimate lower bound for

$$\max_{a_i \leq \delta_i \leq b_i} \tilde{\sigma} \left[ M_{22} + M_{21} D_\delta (I - M_{11} D_\delta)^{-1} M_{12} \right]$$

Evaluating anywhere in the cube gives a lower bound on the maximum – try to improve this

- Start at center of $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$
- Iterate as follows:
  - Holding $\delta_2, \ldots, \delta_n$ fixed at their “current” values, adjust $\delta_1$ in $[a_1, b_1]$ to maximize $\tilde{\sigma}$
  - Holding $\delta_1, \delta_2, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_n$ fixed at their “current” values, adjust $\delta_i$ in $[a_i, b_i]$ to maximize $\tilde{\sigma}$
  - and so on, cycling back and forth through the $\delta$’s

**Issues:**

- Order of cycling (in general) affects final convergence
- Initial starting point need not be the center, and (in general) affects final convergence
- No guarantee that iteration converges to maximum, but it always does at least as good as taking the value at the center.
- For a fixed $M$, as the width of the intervals go to zero, this “bounding” technique gets the right answer.
- How do we maximize over each individual $\delta_i$?
With all but one of the δ’s held fixed, the problem appears as (with different M, depending on values of parameters being held fixed)

**Problem:** Given $M \in C^{(k+n_e) \times (k+n_d)}$. Solve

$$\max_{-1 \leq \delta \leq 1} \bar{\sigma} \left[ M_{22} + M_{21} \delta (I - \delta M_{11})^{-1} M_{12} \right]$$

Mimicking Hamiltonian methods for state-space $\mathcal{H}_\infty$ norm ...

**Lemma:** Take $\gamma > \bar{\sigma} (M_{22})$. If there is a $\delta_0 \in [-1, 1]$ such that $F_u(M, \delta_0 I_k)$ has a singular value equal to $\gamma$, then the matrix $H_\gamma$

$$\begin{bmatrix}
M_{11} & M_{12} M_{12}^* \\
0 & M_{11}^*
\end{bmatrix} + \begin{bmatrix}
M_{12} M_{22}^* \\
M_{22}^*
\end{bmatrix} (\gamma^2 I - M_{22} M_{22}^*)^{-1} \begin{bmatrix}
M_{21} & M_{22} M_{12}^*
\end{bmatrix}$$

has a real eigenvalue $\lambda$ satisfying $|\lambda| \geq 1$ (specifically, $\delta_0 = \frac{\frac{k}{\lambda}}{}$).

**How:**

$F_u(M, \delta_0 I_k)$ has a singular value equal to $\gamma$

$\Downarrow$

$\gamma^2 I - F_u(M, \delta_0 I_k) \left[ F_u(M, \delta_0 I_k) \right]^* \text{ is singular}$

$\Downarrow$

$K_\gamma(\delta) := \left[ \gamma^2 I - F_u(M, \delta I_k) \left[ F_u(M, \delta I_k) \right]^* \right]^{-1} \text{ has a pole at } \delta = \delta_0$

Poles of $K_\gamma(\delta)$ are (subset of) reciprocals of eigenvalues of $H_\gamma$.

**Remarks:**

- The real eigenvalues of $2k \times 2k$ complex matrix $H_\gamma$ give limited information about the sublevel sets of $f(\delta) := \bar{\sigma} \left[ F_u(M, \delta I_k) \right]$.
- Iterative algorithm to bound maximum by repeatedly computing the eigenvalues of $H_\gamma$ for increasing $\gamma$.
- Controllability/Observability assumptions on $(M_{11}, M_{12})$ and $(M_{11}, M_{21})$ render the theorem necessary and sufficient.
Algorithm: choose relative stopping tolerance $\epsilon > 0$.

1. Set $\tilde{\gamma} := \max \{\tilde{\sigma} [F_u (M, -I)] , \tilde{\sigma} [F_u (M, 0)] , \tilde{\sigma} [F_u (M, I)] \}$. Let $\bar{p}$ be the maximizer from \{-1, 0, 1\}.

2. Define $\gamma := (1 + \epsilon) \tilde{\gamma}$. Form $H_{\gamma}$ and compute eigenvalues.

3. If there are no real eigenvalues with magnitude $\geq 1$, STOP. Bounds are $\tilde{\gamma} \leq \max_{-1 \leq \delta \leq 1} \tilde{\sigma} (\cdot) < (1 + \epsilon) \tilde{\gamma}$, with lower bound achieved by $\delta := \bar{p}$.

4. If there are any real eigenvalues, with magnitude $= 1$, denote their reciprocals as $\{r_i\}_{i=1}^{N}$.

5. Let $\{p_i\}_{i=1}^{N-1}$ denote the midpoints, $p_i := \frac{1}{2} (r_i + r_{i+1})$.

6. Redefine $\bar{p}$ to be the maximizer below

$$\tilde{\gamma} := \max_{1 \leq i \leq N-1} \tilde{\sigma} [M_{22} + M_{21} p_i (I - p_i M_{11})^{-1} M_{12}]$$

7. Go to step 2
With all but one of the δ’s held fixed, the problem appears as (with different \(M\), depending on values of parameters being held fixed)

**Problem:** Given \(M \in \mathbb{C}^{(k+1) \times (k+1)}\), and \(a < b\), find

\[
\max_{a \leq \delta \leq b} |F_u(M, \delta I_k)|
\]

Dependence is rational, namely

\[
F_u(M, \delta I_k) = m_{22} + m_{21} \delta (I - M_{11} \delta)^{-1} m_{12} = \frac{n(\delta)}{d(\delta)}
\]

where

- \(n\) and \(d\) are \(k\)’th order polynomials
- coefficients of \(n\) and \(d\) are complex (since \(M\) is)
- \(n\) and \(d\) are easily computed from \(M\)

Write \(n(\delta) = f(\delta) + jg(\delta)\) where \(f\) and \(g\) are real and imaginary parts. Similar for \(d(\delta) = h(\delta) + jq(\delta)\). Note that

\[
\left| \frac{n(\delta)}{d(\delta)} \right|^2 = \frac{f^2 + g^2}{h^2 + q^2}
\]

is differentiable, and has critical points (slope equal 0) at same locations as \(|F_u(M, \delta I_k)|\).
Task: Find $\delta \in [a, b]$ where either

$$\frac{d}{d\delta} \left( \frac{n(\delta)^2}{d(\delta)} \right) = 0, \quad \text{or} \quad d(\delta) = 0$$

These are precisely the roots of the polynomial

$$c := [ff' + gg'] \left( h^2 + q^2 \right) - [hh' + qq'] \left( f^2 + g^2 \right) = 0.$$ 

which is order $4k - 2$.

Procedure:

1. Compute $n$ and $d$ from $M$
2. Form $c(\delta)$, and compute roots
3. Evaluate $F_u(M, \delta I_k)$ at $a$, $b$ and all real roots in interval $(a, b)$
4. Maximum value of $|F_u(M, \delta I_k)|$ occurs at one of these
Power Method

For just complex uncertainties,

$$\Delta := \{ \text{diag} [\Delta_1, \ldots, \Delta_f] : \Delta_i \in \mathbb{C}^{m_i \times m_i} \}$$

a power-method works well for worst-case gain. Assume $$(I - M_{11} \Delta)$$ is nonsingular for all $$\Delta \in B_\Delta$$. If

$$\max_{\Delta \in B_\Delta} \bar{\sigma} \left[ M_{22} + M_{21} \Delta (I - M_{11} \Delta)^{-1} M_{12} \right] =: \gamma^2$$

(+ technical conditions) then there exist unit-vectors $$a, b, z$$ and $$w$$

$$a = M_\gamma b$$
$$z_1 = \frac{w_1}{a_1} a_1, \ldots, z_f = \frac{w_f}{a_f} a_f, \quad z_{f+1} = \frac{w_{f+1}}{a_{f+1}} a_{f+1}$$
$$w = M_\gamma^* z$$
$$b_1 = \frac{a_1}{w_1} w_1, \ldots, b_f = \frac{a_f}{w_f} w_f, \quad b_{f+1} = \frac{a_{f+1}}{w_{f+1}} w_{f+1}$$

where

$$M_\gamma := \begin{bmatrix} H_{1,1} & \cdots & H_{1,f} & \frac{1}{\gamma} H_{1,f+1} \\ \vdots & \ddots & \vdots & \vdots \\ H_{f,1} & \cdots & H_{f,f} & \frac{1}{\gamma} H_{f,f+1} \\ \frac{1}{\gamma^2} H_{f+1,1} & \cdots & \frac{1}{\gamma^2} H_{f+1,f} & \frac{1}{\gamma^2} H_{f+1,f+1} \end{bmatrix} =: \begin{bmatrix} M_{11} & \frac{1}{\gamma} M_{12} \\ \frac{1}{\gamma} M_{11} & \frac{1}{\gamma^2} M_{22} \end{bmatrix}$$

Try to find solutions by iterating, in the order written. Two facts about solutions (existence and what they mean)

- Any solution $$(a, b, w, z, \gamma)$$ yields a $$\Delta \in B_\Delta$$ achieving a gain of (at least) $$\gamma$$
- The maximum achievable gain is always a solution
Parametrize a class of matrices for which, by construction,

$$\max_{a_i \leq \delta_i \leq b_i} \sigma \left[ M_{22} + M_{21} D_{\delta} (I - M_{11} D_{\delta})^{-1} M_{12} \right] = 1$$

Run algorithm – again, just

1. Hold complex uncertainties ($\Delta_j$) fixed, maximize over real parameters using coordinate-wise ascent
2. Holding real parameters ($\delta_i$) fixed, maximize over complex uncertainties using power method
3. Repeat a few times

Compare answer obtained to 1. Extrapolation of performance to other matrices is not clear, but this a starting benchmark.

3 “cases,” 300 examples/case

- $2 \leq n \leq 12$, $k_i = 1$ for all $i \leq n$; $f = 2$ with $k_{n+1} = k_{n+2} = 1$, performance dimensions $n_d = n_e = 2$.
- $2 \leq n \leq 12$, $k_i = 3$ for all $i \leq n$; $f = 2$ with $k_{n+1} = k_{n+2} = 1$, performance dimensions $n_d = n_e = 2$.
- $6 \leq n \leq 25$, $k_i = 4$ for all $i$; $f = 0$ (no additional uncertainty blocks), performance dimensions $n_e = n_d = 4$.

Flop-count (averaged over the 300 runs) for the last case. Appears to get polynomial growth in the computational effort with respect to number-of-uncertainties.
All of the results compare favorably to those in [6], with the exception of the flop-count, which is not discussed in [6]. It is likely that those methods involve less computations than ours.

Main drawback is that a very explicit representation of $M$ is required, since $H_\gamma$ must be formed, and real eigenvalues computed. This is in contrast to the mixed (real & complex) power-method, [6], [7], [15], [16], [17], where only algorithms that compute $Mx$ and $M^*v$, given $x$ and $v$, are needed.
Again, parametrize a class of matrices for which, by construction,
\[
\max_{a_i \leq \delta_j \leq b_j, \sigma(\Delta_i) \leq 1} \bar{\sigma} \left( M_{22} + M_{21} D_{\delta \Delta} (I - M_{11} D_{\delta \Delta})^{-1} M_{12} \right) = 1
\]

Run algorithm, compare answer obtained to 1. Extrapolation of performance to other matrices is not clear, but this a starting benchmark.

2 cases, 300 examples/case

- \( n = 0, 2 \leq n \leq 15, k_i = 1 \) for all \( i \), performance dimensions \( n_d = n_e = 2 \).

- \( n = 0, 2 \leq n \leq 15, k_i = 2 \) for all \( i \), performance dimensions \( n_d = n_e = 2 \).

But, there are some cases for which convergence is not as nice...
Power Method (Complex)  More Numerical Tests

One additional case:

\( n = 0, \, 2 \leq n \leq 15, \, k_i = 2 \) for all \( i \), but \( \Delta_i = \delta_i I_2 \) instead of \( 2 \times 2 \) full matrix.

Performance dimensions \( n_d = n_e = 2 \).
A few observations at this point

- Based on comparisons with [6], the coordinate-wise approach to worst-case gain with real parameter uncertainties is favorable to the mixed (real/complex) power algorithm.

- Complex uncertainties can also be individually maximized over. Interestingly, we have found this approach to be inferior to existing complex power methods.

- “Coordinate-wise” across the group of real parameters and the group of complex uncertainties is adequate for now, but could be improved...
Suppose real parameters only (for notation): associated with dimensions $k_1, k_2, \ldots, k_n$, define sets of matrices 

$$D := \{ \text{diag}[D_1, D_2, \ldots, D_n] : 0 < D_i = D_i^* \in \mathbb{C}^{k_i \times k_i} \}$$

and 

$$G := \{ \text{diag}[G_1, G_2, \ldots, G_n] : G_i = -G_i^* \in \mathbb{C}^{k_i \times k_i} \}$$

Applications of the $S$-procedure (separating hyperplane) yields:

**Theorem:** If there is an $D \in D, G \in G$ and $\beta > 0$ such that 

$$A(M, X, G, \beta^2) := 
\begin{bmatrix}
  D & 0 \\
  0 & \beta^2 I \\
\end{bmatrix} - M^* 
\begin{bmatrix}
  D & 0 \\
  0 & I \\
\end{bmatrix} M + M^* 
\begin{bmatrix}
  G & 0 \\
  0 & 0 \\
\end{bmatrix} - 
\begin{bmatrix}
  G & 0 \\
  0 & 0 \\
\end{bmatrix} M \geq 0 
$$

then 

$$\max_{-1 \leq \delta_i \leq 1} \bar{\sigma}[F_u(M, D_\delta)] \leq \beta$$

Get best bound via minimization of $\gamma := \beta^2$, subject to the constraints 

$$D \in D, G \in G, \gamma > 0, A(M, X, G, \gamma) \geq 0$$

Useful properties 

- Linear objective, over the variable $(D, G, \gamma)$
- Convex constraints

**Question:** When repeating the computation on a subdivided cube, can anything be re-used from the original cube’s computation?
**Redheffer Star Products**

**Definition**

$T$ and $B$ are compatibly partitioned matrices, with $T_{22}B_{11}$ well-defined, and square.

Consider the constraints

$$
\begin{bmatrix}
y_1 \\
z
\end{bmatrix} = T
\begin{bmatrix}
u_1 \\
w
\end{bmatrix}, \quad
\begin{bmatrix}
w \\
y_2
\end{bmatrix} = B
\begin{bmatrix}
z \\
u_2
\end{bmatrix}
$$

drawn as

![Diagram](attachment:image.png)

**Fact:** For each $u_1, u_2$, there exist unique vectors $z, w, y_1$ and $y_2$ solving the constraints if and only if $\det(I - T_{22}B_{11}) \neq 0$.

In that case, the “star product $(T*B)$ is well-posed,” and $T*B$ is defined as the $2 \times 2$ block matrix relating the $u_i$ to the $y_i$. 
**Lemma:** Suppose $T \in \mathbb{C}^{(n_1+n_2) \times (n_1+n_1)}$ and $M \in \mathbb{C}^{(n_1+n_2) \times (n_1+n_1)}$ are compatibly partitioned matrices, with $I - T_{22}M_{11}$ invertible. Assume $T_{21}$ is invertible, and $g \in \mathbb{C}$, with $\text{Re}(g) = 0$. If

$$M^*M + g(M - M^*) < I$$

and

$$T^*T + g(T - T^*) \leq I,$$

then

$$(T\star M)^*(T\star M) + g[(T\star M) - (T\star M)^*] < I$$

For $g = 0$, this is just: If

$$M^*M < I \quad \text{equivalently} \quad \bar{\sigma}(M) < 1$$

and

$$T^*T \leq I \quad \text{equivalently} \quad \bar{\sigma}(T) \leq 1,$$

then

$$(T\star M)^*(T\star M) < I \quad \text{equivalently} \quad \bar{\sigma}(T\star M) < 1$$

which is an easy case.
Again: $T_{21}$ is invertible, $T \ast M$ well-posed, $g$ purely imaginary,

$$M^* M + g (M - M^*) < I$$

and

$$T^* T + g (T - T^*) \leq I.$$  

Then

$$(T \ast M)^* (T \ast M) + g [(T \ast M) - (T \ast M)^*] < I$$

**Proof:** By assumption, $T \ast M$ is well-posed. Let $u_i \in C^{n_i}$ be arbitrary, not both 0. Let $y_i$ and $z$ and $w$ be the unique solutions to the defining star-product equations

$$
\begin{bmatrix}
    y_1 \\
    z 
\end{bmatrix} = T
\begin{bmatrix}
    u_1 \\
    w 
\end{bmatrix}, \\
\begin{bmatrix}
    w \\
    y_2 
\end{bmatrix} = M
\begin{bmatrix}
    z \\
    u_2 
\end{bmatrix}
$$

Since $T_{21}$ is invertible, it follows that $u_2$ and $z$ are not both zero.

The two hypothesis each combine with the star-product constraints to respectively give

$$w^* w + y_2^* y_2 + g [(z^* w + u_2^* y_2) - (w^* z + y_2^* u_2)] < z^* z + u_2^* u_2$$

$$y_1^* y_1 + z^* z + g [(u_1^* y_1 + w^* z) - (y_1^* u_1 + z^* w)] \leq u_1^* u_1 + w^* w$$

Adding these, and cancelling leaves $y^* y + g (u^* y - y^* u) < u^* u$, which, since $u$ was arbitrary, and $y = (T \ast M) u$, implies the desired conclusion. 

**Remarks:** Suppose $T$ satisfies $\bar{\sigma}(T) \leq 1$, and $T = T^*$. It is easy to verify that for all imaginary $g$, $T$ satisfies the hypothesis. Moreover, if $T_{21}$ is not invertible, then both $<$ are changed to $\leq$. 

27
Suppose \( T \) and \( M \) given, with \( I - T_{22}M_{11} \) invertible. \( T_{21} \) is invertible. Quantities \( \beta > 0, D = D^* > 0 \) and \( G = -G^* \) are also given.

If
\[
M^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} M - M^* \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}
\]
and
\[
T^* \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} T + \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} T - T^* \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \leq \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}
\]
then
\[
(T^*M)^* \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} (T^*M) + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} (T^*M) - (T^*M)^* \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} < \begin{bmatrix} D & 0 \\ 0 & \beta^2 I \end{bmatrix}
\]

**Proof:** Extension of the previous result.

**Remark:** Suppose that \( T \) satisfies \( \sigma(T) \leq 1, T = T^* \), and
\[
GT_{ij} = T_{ij}G, \quad D^{1/2}T_{ij} = T_{ij}D^{1/2}
\]
for \( 1 \leq i \leq 2, 1 \leq j \leq 2 \). Then \( T \) satisfies
\[
T^* \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} T + \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} T - T^* \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \leq \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}
\]
(the main hypothesis).

If \( T_{21} \) is not invertible, then both \( < \) are changed to \( \leq \).
Well-posed star products are associative.

Suppose that $T, M$ and $B$ are compatibly partitioned matrices, Assume that $T \ast M$ is well-posed, and that $M \ast B$ is well-posed.

Then

$$(T \ast M) \ast B \text{ is well posed } \iff T \ast (M \ast B) \text{ is well posed}$$

Under these conditions,
For a given pair $a$ and $b$, representing the cube

$$Q_{[a,b]} := [a_1 b_1] \times [a_2 b_2] \times \cdots \times [a_n b_n]$$

define “center” and “radius” matrices

$$C_{[a,b]} := \begin{bmatrix} \frac{b_1-a_1}{2} I_{k_1} & 0 & \cdots & 0 \\ 0 & \frac{b_2-a_2}{2} I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{b_n-a_n}{2} I_{k_n} \end{bmatrix}$$

and

$$R_{[a,b]} := \begin{bmatrix} \frac{b_1-a_1}{2} I_{k_1} & 0 & \cdots & 0 \\ 0 & \frac{b_2-a_2}{2} I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{b_n-a_n}{2} I_{k_n} \end{bmatrix}$$

Define $M_{CR}$ as below

$$M_{CR} = C M R^{1/2}$$

Clearly

$$\max_{a_i \leq \delta_i \leq b_i} \bar{\sigma} [F_u (M, \mathcal{D}_\delta)] = \max_{-1 \leq \xi_i \leq 1} \bar{\sigma} [F_u (M_{CR}, \mathcal{D}_\xi)]$$

Recentering/Normalizing: consider the unit cube when useful
Suppose two cubes are given, by vectors $a, b$ and $\tilde{a}, \tilde{b}$. Associated with each, define center and radius matrices, $C, \tilde{C}, R$ and $\tilde{R}$. How do we transform from $M_{CR} \rightarrow M_{\tilde{C}R}$?

Start with $M_{CR}$

Cancel $C$ and $R$, replacing with $\tilde{C}$ and $\tilde{R}$

leaving $M_{\tilde{C} \tilde{R}}$
Define matrix $T$ as

$$T := \begin{bmatrix} 0 & \tilde{R}_{\frac{1}{2}} R^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \tilde{R}_{\frac{1}{2}} & R^{-\frac{1}{2}} (\tilde{C} - C) R^{-\frac{1}{2}} \end{bmatrix}$$

Block diagram of $T$ is

Through the star-product, $T$ relates $M_{CR}$ to $M_{\tilde{C}\tilde{R}}$

leaving $M_{\tilde{C}\tilde{R}} = T \ast M_{RC}$
Is $Q_{[\tilde{a}, \tilde{b}]} \subset Q_{[a,b]}$?

Condition on $T$

The quantity $\tilde{\sigma}(T)$ determines whether the cube defined by $(\tilde{C}, \tilde{R})$ is contained in the cube defined by $(C, R)$.

The scalar version is:

**Lemma:** Given $c, \tilde{c} \in \mathbb{R}$, and $r > 0, \tilde{r} \geq 0$. Then

$$c - r \leq \tilde{c} - \tilde{r}, \text{ and } \tilde{c} + \tilde{r} \leq c + r$$

if and only if

$$\tilde{\sigma} \begin{bmatrix} 0 & \sqrt{\frac{\tilde{r}}{r}} \\ \sqrt{\frac{r}{\tilde{r}}} & \frac{\tilde{c} - c}{r} \end{bmatrix} \leq 1.$$

Hence, $Q_{[\tilde{a}, \tilde{b}]} \subset Q_{[a,b]}$, if and only if the associated $T$

$$T = \begin{bmatrix} 0 & \tilde{R}^\frac{1}{2} R^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \tilde{R}^\frac{1}{2} & R^{-\frac{1}{2}} (\tilde{C} - C) R^{-\frac{1}{2}} \end{bmatrix}$$

satisfies $\tilde{\sigma}(T) \leq 1$. In any case, $T = T^*$.

Moreover, the structure of $D$ and $G$ imply that for $i, j = 1, 2$

$$GT_{ij} = T_{ij} G, \quad D^{1/2}T_{ij} = T_{ij} D^{1/2}$$

Combining all of these ideas yields the desired result.
**Theorem:** Given $M$ and two cubes $Q_{[a,b]}$ and $Q_{[\tilde{a},\tilde{b}]}$. Let $R$, $C$, $\tilde{R}$ and $\tilde{C}$ be the associated “radius/center” matrices. Assume that $I - M_{11}C$ invertible.

If there exist $D \in D$, $G \in G$ and $\beta > 0$ such that

$$M^*_R [D \ 0 \ 0] M_{RC} + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} M_{RC} - M^*_r [G \ 0] < [D \ 0 \ 0]$$

and $Q_{[\tilde{a},\tilde{b}]} \subset Q_{[a,b]}$, then also

$$M^*_\tilde{R} [D \ 0 \ 0] M_{\tilde{R}C} + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} M_{\tilde{R}C} - M^*_\tilde{r} [G \ 0] < [D \ 0 \ 0]$$

**Implication:** When subdividing a cube in Divide-and-Conquer scheme,

- the decision variables $(D, G, \beta)$ obtained in the previous optimization are feasible for subdivided cube.
- hence, the optimization need not first obtain feasibility.

This seems to save about 25-30% in total upper bound computation.
- Bounds are valid for cube aligned with \((\delta_1, \delta_2, ..., \delta_n)\) axis
- In X-38 model, two aerodynamic coefficients, \(CL_{dr}\) and \(CN_{dr}\) are correlated, and their uncertainties, \(\delta_{CL_{dr}}\) and \(\delta_{CN_{dr}}\) are correlated, assumed to lie in region below

\[
\delta_{CN_{dr}} \quad \delta_{CL_{dr}}
\]

- Create LFT functions that approximately map a unit cube into such a region, for example

\[
\begin{array}{c}
n_2 \\
\end{array}
\quad
\begin{array}{c}
n_1 \\
\end{array}
\]

into

\[
\begin{array}{c}
\delta_{CN_{dr}} \\
\end{array}
\quad
\begin{array}{c}
\delta_{CL_{dr}} \\
\end{array}
\]

or

\[
\begin{array}{c}
\delta_{CN_{dr}} \\
\end{array}
\quad
\begin{array}{c}
\delta_{CL_{dr}} \\
\end{array}
\]
Correlated Parameters

Some formulae which approximately map the cube \((n_1, n_2)\) to \((\delta_1, \delta_2)\)

\[
\delta_{CLdr} = -\frac{n_1}{\sqrt{2}} \left( \sqrt{2} - 2\beta \frac{n_2^2}{1 + n_2^2} \right) + \frac{\beta n_2}{\sqrt{2}}
\]

\[
\delta_{CNdr} = \frac{n_1}{\sqrt{2}} \left( \sqrt{2} - 2\beta \frac{n_2^2}{1 + n_2^2} \right) + \frac{\beta n_2}{\sqrt{2}}
\]

where \(n_1\) and \(n_2\) each independently range from \([-1,1]\). An example, with \(\beta = \frac{1}{2\sqrt{2}}\) is shown below.
How is this represented as an LFT? Define $5 \times 5$ matrices $S_l, S_n$

\[
S_l := \begin{bmatrix}
0 & -\beta \sqrt{2} & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{\beta}{\sqrt{2}} \\
-1 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad S_n := \begin{bmatrix}
0 & -\beta \sqrt{2} & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{\beta}{\sqrt{2}} \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Then for any $n_1, n_2$, letting $N$ denote

\[
N := \begin{bmatrix}
n_1 & 0 & 0 & 0 \\
0 & n_2 & 0 & 0 \\
0 & 0 & n_2 & 0 \\
0 & 0 & 0 & n_2
\end{bmatrix}
\]

gives

\[
F_u(S_l, N) = -\frac{n_1}{\sqrt{2}} \left( \sqrt{2} - 2\beta \frac{n_2^2}{1 + n_2^2} \right) + \frac{\beta n_2}{\sqrt{2}}
\]

and

\[
F_u(S_n, N) = \frac{n_1}{\sqrt{2}} \left( \sqrt{2} - 2\beta \frac{n_2^2}{1 + n_2^2} \right) + \frac{\beta n_2}{\sqrt{2}}
\]

So: replace every instance of $\delta_{CL_{dr}}$ with $F_u(S_l, N)$, and every instance of $\delta_{CN_{dr}}$ with $F_u(S_n, N)$

**Observation:** Nice approximation to region, but LFT representation involves 6 copies of $n_2$ for every copy of $\delta_{CL_{dr}}$ and $\delta_{CN_{dr}}$ in the original problem.
Simpler, though cruder approximation to map the cube \((n_1, n_2)\) to desired region is

\[
\begin{align*}
\delta_{CL_{dr}} &= -n_1 - \frac{1}{4}n_2 \\
\delta_{CN_{dr}} &= n_1 - \frac{1}{4}n_2
\end{align*}
\]

where \(n_1\) and \(n_2\) each independently range from \([-1, 1]\). This is shown below.
Results

<table>
<thead>
<tr>
<th>No correlation constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{CN_{dr}}$</td>
</tr>
<tr>
<td>$\delta_{CL_{dr}}$</td>
</tr>
<tr>
<td>m44h26</td>
</tr>
<tr>
<td>$2.88 \leq \text{WCP} \leq 2.88$</td>
</tr>
<tr>
<td>jscm44h25t10</td>
</tr>
<tr>
<td>$3.84 \leq \text{WCP} \leq 3.86$</td>
</tr>
</tbody>
</table>

| m7h42t10                  |
| $4.26 \leq \text{WCP} \leq 4.32$ |
**Results**

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

m44h26

\[ 2.06 \leq WCP \leq 2.09 \]

**Rotated Linear Fit**

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

jscm44h25t10

\[ 3.98 \leq WCP \leq 3.98 \]

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

jscm7h42t10

\[ 1.91 \leq WCP \leq 1.91 \]

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

m7h42t10

\[ 2.10 \leq WCP \leq 2.11 \]
Results

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

- **m44h26**
  - \( 2.03 \leq \text{WCP} \leq 2.04 \)

- **jscm44h25t10**
  - \( 3.84 \leq \text{WCP} \leq 3.84 \)

Rational Fit

\[ \delta_{CN_{dr}} \rightarrow \delta_{CL_{dr}} \]

- **jscm7h42t10**
  - \( 1.82 \leq \text{WCP} \leq 1.82 \)

- **m7h42t10**
  - \( 1.80 \leq \text{WCP} \leq 1.82 \)


